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Wavelength-dependent bulk parameters for coherent sound in correlated distributions of small-spaced scatterers

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Earlier results for coherent propagation of sound in correlated random distributions of two-parameter particles of radius a (with minimum separation $b > 2a$ small compared to wavelength $\lambda = 2\pi/k$) are generalized to obtain the refractive and absorptive terms and the corresponding bulk parameters to order $(ka)^2$. The present development includes higher order terms of the earlier multiple scattering by monopoles and dipoles, as well as scattering and multipole-coupling effects through quadrupole terms. The correlation aspects are determined by the statistical mechanics radial distribution function $f(R)$ for impenetrable particles of diameter b. The new terms for slab scatterers and spheres involve the integral of fR (first moment), or of $Rf\ln R$ for cylinders. The new packing factor is evaluated exactly for slabs as a simple algebraic function of the volume fraction w, and it is shown that the bulk index of refraction reduces to that of one particle in the limit $w = 1$. Similar results are obtained for spheres in terms of the Percus–Yevick approximation and the unrealizable limit $w = 1$.

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INTRODUCTION

We apply general results for coherent propagation in pair-correlated random distributions[1] of particles with minimum separation (b) of centers small compared to wavelength ($\lambda = 2\pi/k$) to obtain additional terms in k of the bulk parameters (C,B) and index of refraction ($\eta^2 = C/B$) considered before.[2] Using γ for either C, B, or η^2, and the form $\gamma = \gamma_r + i\gamma_a + i\gamma_s$, for small kb, we obtained[2] results for the refractive (γ_r) and absorptive (γ_a) terms that were explicitly independent of k, and corresponding results for the scattering (γ_s) loss term to lowest order in k. The explicit approximations for γ_r and γ_a for spheres, cylinders, and slabs ($m = 3, 2, 1$, respectively) depended only on the particles' radius or half-width (a), their acoustic parameters, and on their average number (ρ) per unit volume: They exhibited the statistical aspect of the problem only in the volume fraction $w = \rho v$ with $v = v(a)$ as the volume of one particle. The corresponding scattering terms γ_s were additionally dependent on $(ka)^m$ and on the low-frequency limit of the structure factor $\mathscr{W}(W)$ with $W = w(b/2a)^m$ as the volume fraction of impenetrable statistical particles with diameter $b > 2a$, i.e., in general, each particle was visualized as having an acoustically transparent coating of thickness $(b/2) - a$. The present paper applies the general theory[1] to derive the leading λ-dependent terms of γ_r and γ_a; these depend explicitly on $(ka)^2$ and a/b for all cases, and on appropriate correlation integrals $\mathscr{N}(W)$. We parallel a recent development for the simpler one-parameter optical case.[3]

The correlation aspects of the distribution we consider are determined by the statistical-mechanics radial distribution function[4] $f(R)$ for impenetrable particles, and are exhibited explicitly as simple integrals over all R of the total correlation function $F = f - 1$. The integrals for spherical and slab particles are of the form $\int FR^n\,dR$ (moments of F), but cylindrical particles also involve $\int RF(\ln R)dR$. These can all be evaluated numerically from existing statistical mechanics forms or approximations[4–15] for f. We obtained explicit closed form approximations before[2,16] for the integrals that arise in the $\mathscr{W}$ set, and also used the required $\mathscr{N}$ integral for spheres in a related development[17] for large kb; for slabs, we obtain both the $\mathscr{W}$ and $\mathscr{N}$ integrals from our earlier Laplace transformation[13] of the exact Zernike–Prins result[12] for f. For cylinders, we may use the virial expansion for f to consider some of the properties of $\mathscr{N}$.

In the following, for brevity, we use, for example (1:113) to indicate Eq. (113) of Ref. 1, as well as essentially the same notation as before.[1–3] We generalize the earlier[2] multiple scattering monopole and dipole approximations for C, B, and η^2 by including scattering and multipole coupling to higher orders in k, as well as quadrupoles (for spheres and cylinders) to lowest order. For slabs and $b = 2a$ (minimum separation of slab centers equal to slab thickness), the explicit approximations for the bulk values (γ) reduce to the single particle values (γ') if $w \rightarrow 1$, as required from physical considerations: $\gamma \rightarrow \gamma'$ because the particles occupy all space. The limit $w \rightarrow 1$ is not realizable for identical spheres, and we take $w \lesssim w_d \approx 0.63$, with w_d as the densest random packing introduced earlier[16] to define the amorphous solid. However, our explicit approximations for γ for spheres also reduce to γ' as w increases to 1; we regard this as consistent with the approximations involved in scaled particle[5] and Percus–Yevick[4,6] statistical mechanics theory, and with the closure approximation used in the multiple scattering theory.[1] For cylinders, we take $w \lesssim w_d \approx 0.84$ as before.[18,2] Were an analogous closed form available for the $\mathscr{N}$ integral for this case, we would expect the corresponding approximations for γ to show the same behavior for the nonrealizable limit $w \rightarrow 1$.

The present application of the general theory[1] to larger kb than before[2] plus the recent applications[17] to large kb provide simple forms which explicitly display the functional dependence on all key parameters for many practical applications. Thus, in these ranges of kb, elaborate machine com-

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putations are no longer required, and the results help delineate the fundamental physical processes.

I. GENERAL CONSIDERATIONS

For a slab-region distribution and a normally incident wave $\phi e^{-i\omega t}$ (the excess pressure) we write

$$\phi = e^{ikz}, \quad k = 2\pi/\lambda = 2\pi\eta_e/\lambda_e, \tag{1}$$

with η_e as the index of the embedding medium. The corresponding bulk coherent propagation coefficient is given by

$$K = k\eta_b/\eta_e = k\eta, \quad \eta^2 = C/B, \tag{2}$$

where in the simplest cases, C is the relative compressibility and B^{-1} is the relative mass density. The values are to be expressed in terms of ρ and F for pair correlated particles specified by their isolated scattering amplitudes $g(\hat{r},\hat{z})$. The normalization for g is such that for lossless particles

$$-\operatorname{Re} g(\hat{z},\hat{z}) = -\operatorname{Re} g = \mathscr{M}|g(\hat{r},\hat{z})|^2, \tag{3}$$

with $\mathscr{M}$ as the mean over all directions of observation $\hat{r}$. The corresponding known[19] scattering coefficients a_n are normalized by the form

$$g = \sum a_n; \quad a_n = a_n(C',B';x), \quad x \equiv ka, \tag{4}$$

where

$$C' = \frac{C_p}{C_e}, \quad B' = \frac{B_p}{B_e}; \quad C' - 1 \equiv \mathscr{C}, \quad B' - 1 \equiv \mathscr{B}, \tag{5}$$

with C' and B' as the particle's relative acoustic parameters, such that $\operatorname{Im} C' > 0$ and $\operatorname{Im} B' < 0$ to account for energy losses; $\mathscr{C}$ and $\mathscr{B}$ are the parametric contrasts. In addition to the dependence on the relative parameters and on x (the normalized radius or half-width), the coefficients depend on the dimensionality (m) of the problem.

We obtain results for the bulk relative values $\gamma = \{C,B,\eta^2\}$ in the form

$$\gamma = \gamma_1 + \gamma_c + i\gamma_s = \Gamma_1 + x^2\Gamma_c + ix^m\Gamma_s, \tag{6}$$

where Γ_i (with $i = 1,c,s$) depends on m. The forms for Γ_1 and Γ_s, corresponding to multiple scattering by monopoles and dipoles to lowest order in x for the real and imaginary parts of a_0 and a_1, were discussed before[2] in detail. Now we obtain Γ_c.

From Rayleigh's results for spherical dipoles,[20] the first approximation for sparse uncorrelated distributions corresponds to

$$\eta_R - 1 = -\frac{cg}{2}, \quad c = \frac{i2\rho}{k}, \frac{i4\rho}{k^2}, \frac{i4\pi\rho}{k^3}, \tag{7}$$

where the order for c (and subsequent sets) corresponds to $m = 1, 2, 3$. In Ref. 18 (1975), for lossless particles with small parametric contrast, we multiplied $\operatorname{Im}\eta_R$ by the statistical mechanics packing factor $\mathscr{W}$ to obtain the appropriate η_s for the correlated case. Appropriate forms for C_1 and B_1 for spheres and slabs were given by Maxwell,[21] and for cylinders and spheres by Rayleigh.[22] We obtained γ_1 and γ_s from (2:59), which followed from either the full multiple scattering procedure or an equivalent slab procedure (interface approximation).[1,2] For γ_c the two procedures lead to different results, as illustrated for circular cylinders by (1:101)ff. Now we apply the multiple scattering procedure to obtain Γ_c explicitly for an unbounded distribution. See discussion after (1:102) and (23:37), as well as after (23:50) which considers the impedance; discrepancies may arise for measurements on plane-bounded distributions because of transition region boundary layers.

Rewriting (2:30) in the original form (1:89)ff, we use

$$-\frac{(\eta^2-1)}{c} = \sum P_n \approx P_0 + P_1 + P_2, \tag{8}$$

where P_2 does not arise for slabs. The coefficients P_n are determined by the linear system of algebraic equations (matrix equation),

$$P_n = a_n\,\eta^{2n}\Big(1 + \sum \mathscr{H}_{n\nu}\,\eta^{-n-\nu}P_\nu\Big), \tag{9}$$

with $\mathscr{H}_{n\nu}$ given in terms of the correlation integrals $\mathscr{H}_n$ by (2:21)ff. All cases we consider are covered by

$$\begin{aligned} P_0 &= a_0\Big(1 + P_0\mathscr{H}_0 + \frac{P_1\mathscr{H}_1}{\eta} + \frac{P_2\mathscr{H}_2}{\eta^2}\Big),\\ P_1 &= a_1\eta^2\Big(1 + \frac{P_0\mathscr{H}_1}{\eta} + \frac{P_1\mathscr{H}_{11}}{\eta^2} + \frac{P_2\mathscr{H}_{12}}{\eta^3}\Big),\\ P_2 &= a_2\eta^4\Big(1 + \frac{P_0\mathscr{H}_2}{\eta^2} + \frac{P_1\mathscr{H}_{12}}{\eta^3} + \frac{P_2\mathscr{H}_{22}}{\eta^4}\Big). \end{aligned} \tag{10}$$

Introducing

$$R = \sum P_\nu \Big/ \sum P_n\left(\frac{d_n}{\eta^n}\right), \tag{11}$$

with $d_n = d_n^B - d_n^C$ as in (1:95)–(1:100), we write the corresponding bulk parameters as

$$\begin{aligned}\frac{C-1}{c} &= R\sum\frac{P_n d_n^C}{\eta^n} \approx -P_0\\ &\quad - \frac{x^2}{m(m+2)}\Big(P_0\,\eta'^2\frac{\mathscr{B}}{\mathscr{C}}(1 - cP_0) - m\frac{\mathscr{C}}{\mathscr{B}}P_1\Big),\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\eta^2(B-1)}{c} &= R\sum\frac{P_n d_n^B}{\eta^n}\\ &\approx P_1 - \frac{x^2}{m(m+2)}\Big(P_0\,\eta'^2\frac{\mathscr{B}}{\mathscr{C}}(\eta^2 + cP_1)\\ &\quad - m\frac{\mathscr{C}}{\mathscr{B}}P_1\Big) + P_2, \quad \eta'^2 = C'/B'.\end{aligned} \tag{13}$$

The sets of d's are expressed in terms of Bessel functions of arguments ka and Ka in the forms (1:96)–(1:100); we use the forms in terms of j_n for spheres, in terms of J_n for cylinders, and in terms $\mathscr{J}_0 = \cos$ and $\mathscr{J}_1 = \sin$ for slabs. See cited equations for full details, and (1:101)–(1:107) for illustrations based on cylinders and for comparisons with the interface approximations. As shown in (1:99) and (1:100), the d^C and d^B sets are proportional to $\mathscr{C}$ and $\mathscr{B}$, respectively; thus $C = 1$ if $C' = 1$, and $B = 1$ if $B' = 1$, as required by the theorems (2:13) and (2:14) discussed earlier[23] in detail.

Using (11), we see that the equality $\eta^2(B-1) - (C-1) = 1 - \eta^2$ is satisfied by the rigorous series forms in (13), (12), and (8). To the accuracy indicated for (6), we may replace $\eta^2 + cP_1$ by $1 - cP_0$ within the large parentheses of

the approximations; thus, the approximations in (13), (12), and (8) satisfy the equality to the required accuracy.

The isolated scattering coefficients[19] for the slab, $a_n = a'_n/(1-a'_n)$ with a'_n imaginary for lossless scatterers, are given by

$$a'_0 = ix(\mathscr{C} - x^2 T_0)/(1 - x^2 D_0) + O(x^5),$$
$$T_0 = [\mathscr{C}(1+\eta'^2) + 2\mathscr{B}\eta'^2]/6,$$
$$D_0 = (\eta'^2 - 1 - 2\mathscr{C})/2; \tag{14}$$
$$a'_1 = -ix(\mathscr{B} - x^2 T_1)/[1 + \mathscr{B} - x^2 D_1],$$
$$T_1 = [\mathscr{B}(1+\eta'^2) + 2\mathscr{C}]/6, \quad D_1 = (\mathscr{B} + \mathscr{C})/2. \tag{15}$$

For the cylinder, we use $a_0 = a'_0/(1-a'_0)$ and $a_n = a'_n/(1 - a'_n/2)$ in terms of

$$a'_0 = i(\pi x^2/4)(\mathscr{C} - x^2 T_0)/(1 - x^2 D_0) + O(x^6),$$
$$T_0 = [\mathscr{C}(1+\eta'^2) + \mathscr{B}\eta'^2]/8,$$
$$D_0 = (\eta'^2 - 1 + 2\mathscr{C}L)/4 \equiv D'_0 + 2\mathscr{C}L/4, \tag{16}$$

where $L = \ln(2/xc')$ with $c' = 1.781...$;

$$a'_1 = -i(\pi x^2/4)(\mathscr{B} - x^2 T_1)/[1 + \mathscr{B}/2 - x^2 D_1] + O(x^6),$$
$$T_1 = [\mathscr{B}(1+\eta'^2) + 2\mathscr{C}]/8,$$
$$D_1 = [3\mathscr{C} - 2\mathscr{B}(1+2L) + \eta'^2 - 1)]/16$$
$$\equiv D'_1 - 4\mathscr{B}L/16; \tag{17}$$
$$a'_2 = -i\pi(x/2)^4\mathscr{B}/(2+\mathscr{B}) + O(x^6). \tag{18}$$

For the sphere, $a_n = a'_n/[1 - a'_n/(2n+1)]$,

$$a'_0 = i(x^3/3)(\mathscr{C} - x^2 T_0)/(1 - x^2 D_0) + O(x^7),$$
$$T_0 = [\mathscr{C}(1+\eta'^2) + 2\mathscr{B}\eta'^2/3]/10,$$
$$D_0 = (\eta'^2 - 1 + 2\mathscr{C})/6; \tag{19}$$
$$a'_1 = -i(x^3/3)(\mathscr{B} - x^2 T_1)/(1 + \mathscr{B}/3 - x^2 D_1) + O(x^7),$$
$$T_1 = [\mathscr{B}(1+\eta'^2) + 2\mathscr{C}]/10,$$
$$D_1 = [3\mathscr{C} - 5\mathscr{B} + 2(\eta'^2 - 1)]/30; \tag{20}$$
$$a'_2 = -ix^5 2\mathscr{B}/9(5 + 2\mathscr{B}) + O(x^7). \tag{21}$$

Although alternative decompositions may be constructed to the same accuracy, the present sets of T's and D's help delineate the dependence on $\mathscr{C} = C' - 1$ and $\mathscr{B} = B' - 1$; comparison of a given set for $m = 1, 2, 3$ indicates the role of dimensionality. If $\mathscr{B} = 0$, then the monopole a_0 dominates; if $\mathscr{C} = 0$, then the dipole a_1 dominates. The ratios of parametric contrasts in (12) and (13) do not introduce singularities; the explicit approximations displayed in the following show that the ratios insure fulfillment of theorems (2:13) and (2:14), so that the development is consistent to the required accuracy. For $m = 2$ and 3, cylinders and spheres, we also consider η^2 for rigid particles ($C' \to 0$, $B' \to 0$) as the limit for $\mathscr{C} \to -1$ and $\mathscr{B} \to -1$.

Using (8)–(21) to lowest orders in x of the real and imaginary parts of all components, we reconstruct (2:63), (2:67), and (2:70) in the present forms γ_1 and γ_s corresponding to (6) and use the results to simplify the subsequent forms for $\gamma_c = x^2 \Gamma_c$. We have

$$C_1 = 1 + w\mathscr{C};$$
$$B_1 = 1 + \frac{w\mathscr{B}}{\mathscr{D}}, \quad \mathscr{D} = 1 + \frac{\mathscr{B}(1-w)}{m}, \tag{22}$$

and

$$\eta_1^2 = C_1/B_1 = (1 + w\mathscr{C})/(1 + w\mathscr{B}/\mathscr{D}), \tag{23}$$

where $\mathscr{C} = C' - 1$ and $\mathscr{B} = B' - 1$ are complex if absorption is present. In distinction to the notation in Ref. 2, the present set γ_1 includes absorption losses, such that $\operatorname{Im} C_1 > 0$, $\operatorname{Im} B_1 < 0$, and $\operatorname{Im} \eta_1^2 = 2 \operatorname{Im} \eta_1 \operatorname{Re} \eta_1 > 0$. The corresponding scattering terms are

$$C_s = x^m d_m \mathscr{C}^2 w \mathscr{W}_m, \quad B_s = -x^m \frac{d_m \mathscr{B}^2}{m\mathscr{D}^2} w \mathscr{W}_m;$$
$$d_m = \{1, \pi/4, \tfrac{1}{3}\}, \tag{24}$$

with packing factors as in (2:33),

$$\mathscr{W}_m = \frac{(1-W)^{m+1}}{[1 + (m-1)W]^{m-1}}. \tag{25}$$

For small absorption, i.e., for $|\operatorname{Re}\gamma'| \gg |\operatorname{Im}\gamma'|$, we use $\mathscr{C}^2 \approx |\mathscr{C}|^2$ and $(\mathscr{B}/\mathscr{D})^2 \approx |\mathscr{B}/\mathscr{D}|^2$.

From (2:70)

$$\frac{C_1 + iC_s}{B_1 + iB_s} \approx \eta_1^2 \left[1 + i\left(\frac{C_s}{C_1} - \frac{B_s}{B_1}\right)\right] \equiv \eta_1^2 + i\eta_s^2;$$
$$\eta_s^2 = \frac{C_s - \eta_1^2 B_s}{B_1} = \frac{x^m d_m w \mathscr{W}}{B_1}\left(\mathscr{C}^2 + \frac{\mathscr{B}^2}{m\mathscr{D}^2}\eta_1^2\right), \tag{26}$$

where $\operatorname{Re}(\eta_s^2/2\eta_1)$ corresponds to attenuation via incoherent scattering losses. Similarly for the present generalization based on (6), from analogous decomposition of $\eta^2 = C/B$, we obtain the additional relation

$$\eta_c^2 = \eta_1^2\left(\frac{C_c}{C_1} - \frac{B_c}{B_1}\right) = \frac{C_c - \eta_1^2 B_c}{B_1}, \tag{27}$$

which provides a check for the independent derivation of the new terms γ_c.

II. DISTRIBUTION OF SLABS

For slabs, we use $c = i2\rho/k = iw/x$, $w = \rho 2a$, and retain only P_0 and P_1 in (8)–(13) with $\mathscr{H}_{11} = c + \mathscr{H}_0$, and $\mathscr{H}_0$ and $\mathscr{H}_1$ as in (1:177) or (2:23). The algebraic system for P_0 and P_1 is valid for all $ka = x$, but we consider only the forms (14) and (15) for a_n and the corresponding leading terms of the correlation integrals

$$\mathscr{H}_0 \approx 2\rho \int_0^\infty F\,dR + ik\,2\rho \int_0^\infty FR\,dR \equiv \mathscr{W} - 1 - ixN,$$
$$\mathscr{H}_1 \approx ix\eta N, \tag{28}$$

where the next terms are $O(x^2)$. We use the rigorous functions[3]

$$\mathscr{W} = (1-W)^2, \quad N = \left(\frac{b}{a}\right)W\left(1 - \frac{4W}{3} + \frac{W^2}{2}\right),$$
$$W = \rho b, \tag{29}$$

with $\mathscr{W} = \mathscr{W}_1$ of (25). For $b = 2a$ corresponding to $W = w$ (minimum separation of particle centers equal to particle width),

$$\mathscr{W}=(1-w)^2,\quad N=2w(1-4w/3+w^2/2). \tag{29'}$$

For full packing $w\to 1$ (the limit of a uniform slab), $\mathscr{W}\to 0$ and $N\to 1/3$. Note that the present N differs in sign from that used earlier[3]; all packing functions are now defined as nonnegative for $w\to 1$.

Solving (10) to the required accuracy in terms of the T's and D's of (14), (15), and $\mathscr{D}=1+\mathscr{B}(1-w)$, we obtain

$$\begin{aligned}
&cP_0=-w(\mathscr{C}-x^2U_0+ix\mathscr{C}^2\mathscr{W});\\
&U_0=T_0-\mathscr{C}D_0-\mathscr{C}^2N-\mathscr{C}\mathscr{B}N\eta^2/\mathscr{D},\\
&T_0-\mathscr{C}D_0=[\mathscr{C}(2-\eta'^2+3\mathscr{C})+\mathscr{B}\eta'^2]/3.
\end{aligned} \tag{30}$$

$$\begin{aligned}
&CP_1=\eta^2w(\mathscr{B}-x^2U_1-ix\mathscr{B}^2\mathscr{W}/\mathscr{D})/\mathscr{D};\\
&U_1=[T_1(1+\mathscr{B})-\mathscr{B}D_1+\mathscr{B}^2N]/\mathscr{D}+\mathscr{C}\mathscr{B}N,\\
&T_1(1+\mathscr{B})-\mathscr{B}D_1=[\mathscr{B}(1-\mathscr{B})+\mathscr{C}]/3.
\end{aligned} \tag{31}$$

Substituting into (8) gives a linear equation for η^2 whose solution plus (30) and (31) also determine C and B of (12) and (13). To lowest orders in x, we obtain the sets $\gamma_1=\Gamma_1$ and $\gamma_s=x\Gamma_s$ as in (22)–(26) for $m=1$. Thus we require only the k^2 correction to γ_1, i.e., $\gamma_c=x^2\Gamma_c$ for $m=1$.

We have

$$\begin{aligned}
&\eta_c^2=\frac{x^2w}{B_1}\left(-U_0+\frac{U_1\eta_1^2}{\mathscr{D}}\right)=\frac{x^2w}{B_1}\left(A_0+\frac{A_1}{\mathscr{D}}\eta_1^2\right),\\
&3A_0\equiv\mathscr{C}[\eta'^2-2-3\mathscr{C}(1-N)]-\mathscr{B}\eta'^2,\\
&3A_1\equiv[\mathscr{B}(1-\mathscr{B}+3\mathscr{B}N)+\mathscr{C}]/\mathscr{D}+6\mathscr{C}\mathscr{B}N,
\end{aligned} \tag{32}$$

$$\begin{aligned}
C_c&=x^2w\left(-U_0+\frac{\mathscr{B}\eta'^2C_1}{3}+\frac{\mathscr{C}\eta_1^2}{3\mathscr{D}}\right)\\
&=\frac{x^2w\,\mathscr{C}}{3}\left(\eta'^2-2-3\,\mathscr{C}(1-N)\right.\\
&\quad\left.+\mathscr{B}\eta'^2w+\eta_1^2\frac{(1+3\mathscr{B}N)}{\mathscr{D}}\right),
\end{aligned} \tag{33}$$

$$\begin{aligned}
B_c&=x^2w\left(-\frac{U_1}{\mathscr{D}}+\frac{\mathscr{B}\eta'^2B_1}{3}+\frac{\mathscr{C}}{3\mathscr{D}}\right)\\
&=\frac{x^2w\,\mathscr{B}}{3}\left(\frac{\mathscr{B}-1-3\mathscr{B}N+\mathscr{C}(1-w)}{\mathscr{D}^2}\right.\\
&\quad\left.-\frac{3\mathscr{C}N}{\mathscr{D}}+\eta'^2B_1\right),
\end{aligned} \tag{34}$$

such that $\eta_c^2B_1=C_c-\eta_1^2B_c$ as required by (27).

If $W=w\to 1$ (the limit of a uniform slab), then $N\to 1/3$ (as well as $\mathscr{D}\to 1$ and $\gamma_1\to\gamma'$), and the set γ_c vanishes. Since $\mathscr{W}\to 0$ for $w\to 1$, the set γ_s also vanishes, and the results for $\gamma=\{\eta^2,C,B\}$ reduce to the values $\gamma'=\{\eta'^2,C',B'\}$ for a single particle as required by elementary physical considerations.

From (23) and (26), plus (32) with η_1^2 replaced by η^2 we reconstruct the linear equation for η^2 that follows directly from (8) in the form

$$\begin{aligned}
\eta^2=&\,1+w(\mathscr{C}+x^2A_0+ix\,\mathscr{C}^2\mathscr{W})\\
&-(w/\mathscr{D})[\mathscr{B}-x^2A_1-ix(\mathscr{B}^2/\mathscr{D})\mathscr{W}]\eta^2.
\end{aligned} \tag{35}$$

The first approximation is $\eta^2=\eta_1^2$ as in (23), and iteration gives $\eta_2^2=\eta_1^2+\eta_c^2+i\eta_s^2$. We proceed similarly in the following sections, where (8) gives a quadratic equation for η^2; in order to minimize elementary algebraic manipulations, the γ_c terms are listed as in (32)–(34), and followed by the equation for η^2 as in (35).

The results simplify for the special cases where either one of the relative parameters equals unity. Thus if $B'=1$, then $\mathscr{B}=0$, $B_c=B_s=0$, and B reduces to unity as required by the general theorem.[23] For this case

$$\eta_c^2=C_c=-x^2w\,\mathscr{C}^2(2-w-3N)/3, \tag{36}$$

with corresponding

$$\eta_1^2=C_1=1+w\,\mathscr{C},\quad \eta_s^2=C_s=xw\,\mathscr{W}\,\mathscr{C}^2. \tag{37}$$

The resulting sum $C=C_1+C_c+iC_s$ is the same form as (3:21) for ϵ in terms of ϵ' and $\delta=\epsilon'-1$.

Similarly, if $C'=1$, then $\mathscr{C}=0$, and C reduces to unity. We have

$$B_c=-\eta_c^2B_1^2=x^2w\,\mathscr{B}^2(2-w-3N)/3\mathscr{D}^2, \tag{38}$$

with corresponding

$$\begin{aligned}
&B_1=\frac{1}{\eta_1^2}=1+\frac{\mathscr{B}w}{\mathscr{D}},\\
&B_s=-\eta_s^2B_1^2=-\frac{xw\,\mathscr{W}\,\mathscr{B}^2}{\mathscr{D}^2}.
\end{aligned} \tag{39}$$

For the present one-dimensional problem, we may reduce the forms for η_i^2 in (38) and (39) to the same forms as in (36) and (37) by introducing $\epsilon'=1/B'$ and identifying $\epsilon'-1$ with $\mathscr{C}$. The general acoustic two-parameter results (32)–(34) also correspond to the general two-parameter (ϵ,μ) electromagnetic cases for $\boldsymbol{\phi}=\hat{\mathbf{x}}\phi$; if $\boldsymbol{\phi}=\mathbf{E}$, then $C'=\epsilon'$ and $B'=1/\mu'$; if $\boldsymbol{\phi}=\mathbf{H}$, then $C'=\mu'$ and $B'=1/\epsilon'$.

For $b=2a$, we define

$$\begin{aligned}
\mathscr{P}&=w(2-w-3N)/2=w(2-7w+8w^2-3w^3)/2\\
&=w(2-3w)(1-w)^2/2,
\end{aligned} \tag{40}$$

where $(1-w)^2=\mathscr{W}$ decreases monotone from 1 to 0 as w increases from 0 to 1. As discussed before, $\mathscr{S}=w\,\mathscr{W}=w(1-w)^2$, which vanishes at $w=0$ and 1, has a maximum at $w=1/3$ corresponding to maximal loss through incoherent scattering (maximal fluctuation effects).[2,18] The function $\mathscr{P}/w$ decreases from unity at $w=0$ to 0 at $w=2/3$, reaches its minimum value at $w=7/9$ and then increases to 0 at $w=1$. The function $\mathscr{P}$ which vanishes at $w=0$, 2/3, and 1, has a maximum at $2w=1-(1/3)^{1/2}$ and a minimum at $2w=1+(1/3)^{1/2}$. The sign of the correction terms γ_c changes at $w=2/3$.

III. DISTRIBUTION OF CYLINDERS

For cylinders, we use $c=i4\rho/k^2=i4w/\pi x^2$, $w=\rho\pi a^2$ in (8)–(13) with

$$\begin{aligned}
&2\mathscr{H}_{11}=c+\mathscr{H}_0+\mathscr{H}_2,\quad 2\mathscr{H}_{12}=c\eta+\mathscr{H}_1+\mathscr{H}_3,\\
&2\mathscr{H}_{22}=c(\eta^2+1)+\mathscr{H}_0+\mathscr{H}_4,
\end{aligned}$$

in terms of the correlation integrals $\mathscr{H}_n$ in (1:71) or (2:22). We have

$$\begin{aligned}
\mathscr{H}_0&\approx 2\pi\rho\int FR\,dR+i4\rho\int F\ln\left(\frac{c'kR}{2}\right)R\,dR\\
&\equiv\mathscr{W}-1+i\mathscr{N},\\
\mathscr{H}_n&\approx-i\left(\frac{\eta^n2\pi\rho}{n\pi}\right)\int FR\,dR=-\frac{i\eta^n(\mathscr{W}-1)}{n\pi},
\end{aligned} \tag{41}$$

where the next terms are $O(x^2)$. We work with

$$\mathscr{W}=1-8W\overline{F}_1;\quad W=\frac{\pi\rho b^2}{4},\quad \overline{F}_1=-\int_0^\infty F(u)u\,du, \tag{42}$$

and

$$-M=L-\frac{\pi\mathscr{N}}{2}=\ln\left(\frac{b}{a}\right)+\mathscr{W}\ln\left(\frac{2}{c'kb}\right)+8W\overline{F}_l;$$

$$\overline{F}_l=-\int_0^\infty F(\ln u)u\,du. \tag{43}$$

Note that the present M differs in sign from that used earlier.[3]

We may evaluate $\overline{F}_1$ and $\overline{F}_l$ numerically by using tabulated values of F, or the original integral equation approximations in the computing routine.[9]

To first order in W, we use the virial expansion[14,3]: $F=-1$ for $u<1$,

$$F=\frac{8W}{\pi}\left\{\cos^{-1}\frac{u}{2}-\frac{u}{2}\left[1-\left(\frac{u}{2}\right)^2\right]^{1/2}\right\},\quad 1<u<2, \tag{44}$$

and $F=0$ for $u>2$. This provides results for $\mathscr{W}$ and M correct to $O(W^2)$, e.g.,

$$\mathscr{W}=1-4W+\sqrt{3}\,12W^2/\pi\simeq1-4W+6.6159W^2. \tag{45}$$

The closed form approximation derived earlier,[18,2] i.e., $\mathscr{W}=\mathscr{W}_2$ of (25),

$$\mathscr{W}=(1-W)^3/(1+W), \tag{46}$$

gives $\mathscr{W}=1-4W+7W^2+\ldots$. For the unrealizable value $W=1$, the closed form $\mathscr{W}$ vanishes; as shown in the following, a comparable approximation of M would reduce to 3/4 for $b=2a$ and $W=1$ in order for γ to equal γ'. The corresponding moments are then $\overline{F}_1=1/8$ and $\overline{F}_l=-(\ln 2)/8-3/32$.

Solving (10) to the required accuracy in terms of a_n and the T's and D's of (16)–(18), and $\mathscr{D}=1+\mathscr{B}(1-w)/2$, we obtain

$$\begin{aligned}
&cP_0=-w(\mathscr{C}-x^2U_0+i\pi x^2\mathscr{C}^2\mathscr{W}/4);\\
&U_0=T_0-\mathscr{C}D_0'\\
&\qquad+\mathscr{C}^2M/2-\mathscr{C}\mathscr{B}(1-\mathscr{W})\eta^2/4\mathscr{D},\\
&T_0-\mathscr{C}D_0'=[\mathscr{C}(3-\eta'^2)+\mathscr{B}\eta'^2/8].
\end{aligned} \tag{47}$$

$$\begin{aligned}
&cP_1=\eta^2w(\mathscr{B}-x^2U_1-i\pi x^2\mathscr{B}^2\mathscr{W}/8\mathscr{D})/\mathscr{D};\\
&U_1=\left(\frac{T_1(2+\mathscr{B})}{2}-\mathscr{B}D_1'-\frac{\mathscr{B}^2M}{4}\right)\frac{1}{\mathscr{D}}\\
&\qquad+\frac{\mathscr{C}\mathscr{B}(1-\mathscr{W})}{4}\\
&\qquad-\frac{\mathscr{B}^2\eta^2}{16}\left(\frac{1-\mathscr{W}}{\mathscr{D}}+w\frac{1+B_1}{1+B'}\right),\\
&\tfrac{1}{2}T_1(2+\mathscr{B})-\mathscr{B}D_1'=\tfrac{1}{16}[\mathscr{B}(3\mathscr{B}+4)+4\mathscr{C}].
\end{aligned} \tag{48}$$

$$\begin{aligned}
&cP_2=\eta^4wx^2U_2,\\
&U_2=\frac{\mathscr{B}(1+B_1)}{8(1+B')}=\frac{\mathscr{B}}{4(2+\mathscr{B})}\left(1+\frac{\mathscr{B}w}{2\mathscr{D}}\right)=\frac{\mathscr{B}}{8\mathscr{D}}.
\end{aligned} \tag{49}$$

The forms for U_2 illustrate other versions that also arise for U_1. Substituting into (8) gives a quadratic equation for η^2 whose solution plus (47)–(49) also determine C and B of (12) and (13). To lowest orders in x, we obtain the sets γ_1 and γ_s as in (22)–(27) for $m=2$, and as before we require only the set $\gamma_c=x^2\Gamma_c$.

We have

$$\begin{aligned}
&\eta_c^2=\frac{x^2w}{B_1}\left(-U_0+\frac{U_1\eta_1^2}{\mathscr{D}}-U_2\eta_1^4\right)=\frac{x^2w}{B_1}\left(A_0+\frac{A_1}{\mathscr{D}}\eta_1^2+A_2\eta_1^4\right);\\
&8A_0=\mathscr{C}(\eta'^2-3-4\mathscr{C}M)-\mathscr{B}\eta'^2,\\
&16A_1=[\mathscr{B}(4+3\mathscr{B}-4\mathscr{B}M)+4\mathscr{C}]/\mathscr{D}+8\mathscr{B}\mathscr{C}(1-\mathscr{W}),\\
&16A_2=-\mathscr{B}^2(1-\mathscr{W})/\mathscr{D}^2-\mathscr{B}[(1+B_1)^2/(1+B')],
\end{aligned} \tag{50}$$

$$C_c=x^2w\left(-U_0+\frac{\mathscr{B}\eta'^2C_1}{8}+\frac{\mathscr{C}\eta_1^2}{4\mathscr{D}}\right)=\frac{x^2w\mathscr{C}}{8}\left(\eta'^2-3-4\mathscr{C}M+\mathscr{B}\eta'^2w+\frac{2\eta_1^2}{\mathscr{D}}[\mathscr{B}(1-\mathscr{W})+1]\right), \tag{51}$$

$$\begin{aligned}
B_c&=x^2w\left(-\frac{U_1}{\mathscr{D}}+\frac{\mathscr{B}\eta'^2B_1}{8}+\frac{\mathscr{C}}{4\mathscr{D}}+\eta_1^2U_2\right)\\
&=\frac{x^2w\mathscr{B}}{16}\left(\frac{\mathscr{B}[-3+4M+\eta_1^2(1-\mathscr{W})]-4+2\mathscr{C}(1-w)}{\mathscr{D}^2}-\frac{4\mathscr{C}(1-\mathscr{W})}{\mathscr{D}}+2\eta'^2B_1+\eta_1^2\frac{(B_1+1)^2}{(B'+1)}\right).
\end{aligned} \tag{52}$$

The set (50)–(52) satisfies $\eta_c^2B_1=C_c-\eta_1^2B_c$ as required by (27). For the unrealizable limit $W=w\to1$, we have $\mathscr{W}=\mathscr{W}_2\to0$ and $\gamma_s=0$, $\gamma_1=\gamma'$; if we use $M\to3/4$, then $\gamma_c\to0$, and $\gamma\to\gamma'$.

The original equation for η^2 obtained directly from (8) by proceeding as for (35) is a quadratic in η^2,

$$\begin{aligned}
\eta^2&=1+w\left(\mathscr{C}+x^2A_0+\frac{i\pi x^2}{4}\mathscr{C}^2\mathscr{W}\right)\\
&\quad-\frac{w}{\mathscr{D}}\left(\mathscr{B}-x^2A_1-\frac{i\pi x^2}{8}\frac{\mathscr{B}^2}{\mathscr{D}}\mathscr{W}\right)\eta^2+wx^2A_2\eta^4.
\end{aligned} \tag{53}$$

The first approximation gives η_1^2, and we obtain $\eta_2^2 = \eta_1^2 + \eta_c^2 + i\eta_s^2$ by iteration.

For the one-parameter case corresponding to $B' = 1$, we have

$$\eta_c^2 = C_c = x^2 w\, \mathscr{C}^2(1 + 2w - 4M)/8, \tag{54}$$

$$\eta_1^2 = C_1 = 1 + w\,\mathscr{C}, \quad \eta_s^2 = C_s = \pi x^2 w\, \mathscr{W}\, \mathscr{C}^2/4. \tag{55}$$

The resulting sum C is the same form as that of ϵ in terms of ϵ' given in (3:26) for the optical case of the electric field parallel ($\mathbf{E}_\parallel$) to the axes of purely dielectric cylinders.

Similarly if $C' = 1$, then

$$\begin{aligned} B_c &= -\eta_c^2 B_1^2 \\ &= -\frac{x^2 w\, \mathscr{B}}{16}\left(\frac{3B' + 1 - \mathscr{B}\,[4M + \eta_1^2(1 - \mathscr{W})]}{\mathscr{D}^2} \right. \\ &\quad \left. - \frac{2B_1}{B'} - \eta_1^2\, \frac{(1 + B_1)^2}{(1 + B')}\right), \end{aligned} \tag{56}$$

$$B_1 = \frac{1}{\eta_1^2} = 1 + \frac{\mathscr{B}\, w}{\mathscr{D}},$$

$$B_s = -\eta_s^2 B_1^2 = -\frac{\pi x^2 w\, \mathscr{W} \mathscr{B}^2}{8\,\mathscr{D}^2}. \tag{57}$$

If we introduce $\epsilon' = 1/B'$ and $\delta = \epsilon' - 1$, then the corresponding sum $\eta^2 = \eta_1^2 + \eta_c^2 + i\eta_s^2 = \epsilon$ is the same as in (3:33) for $\mathbf{E}_\perp$. The general acoustic two-parameter results (50)–(52) correspond to the two-parameter (ϵ,μ) general electromagnetic cases; for $\mathbf{E}_\parallel$, we take $C' = \epsilon'$ and $B' = 1/\mu'$; for $\mathbf{E}_\perp$, we take $C' = \mu'$ and $B' = 1/\epsilon'$.

For comparison with (54) and (55) to $O(\mathscr{B}^2)$, we have

$$B_c \approx -\eta_c^2 \approx -x^2 w\, \mathscr{B}^2(1 + 2w - 4M + \mathscr{W})/16, \tag{58}$$

$$B_1 \approx 1 + w\,\mathscr{B} - w(1 - w)\mathscr{B}^2/2,$$

$$\eta_1^2 \approx 1 - w\,\mathscr{B} + w(1 + w)\mathscr{B}^2/2, \tag{59}$$

$$B_s \approx -\eta_s^2 \approx -\pi x^2 w\, \mathscr{W}\, \mathscr{B}^2/8. \tag{60}$$

The difference in the statistical factors in η_c^2 of (54) and (58) is the packing term $\mathscr{W}$, but both factors vanish for $w = W \to 1$ and $\mathscr{W} \to 0$, $M \to 3/4$.

IV. DISTRIBUTION OF SPHERES

For spheres, we use $c = i4\pi\rho/k^3 = i3w/x^3$, $w = \rho 4\pi a^3/3$ in (8)–(13) with

$$3\,\mathscr{H}_{11} = c + \mathscr{H}_0 + 2\,\mathscr{H}_2, \quad 5\,\mathscr{H}_{12} = 2c\eta + 2\,\mathscr{H}_1 + 3\,\mathscr{H}_3,$$

$$35\,\mathscr{H}_{22} = c(7 + 17\eta^2) + 7\,\mathscr{H}_0 + 10\,\mathscr{H}_2 + 18\,\mathscr{H}_4,$$

in terms of the correlation integrals $\mathscr{H}_n$ of (1:148) or (2:21). We have

$$\mathscr{H}_0 \approx \mathscr{W} - 1 + iN/x, \quad \mathscr{H}_n \approx i\eta^n N/x(2n + 1), \tag{61}$$

where

$$\mathscr{W} = 1 + 4\pi\rho \int_0^\infty FR^2\, dR = 1 - 24W\overline{F}_2;$$

$$W = \frac{4\pi\rho}{3}\left(\frac{b}{2}\right)^3, \quad \overline{F}_2 = -\int_0^\infty F(u)u^2\, du, \tag{62}$$

and

$$N = -4\pi\rho a \int_0^\infty FR\, dR = 24\left(\frac{a}{b}\right) W\overline{F}_1, \tag{63}$$

with the first moment as in (42). The next terms are $O(x)$.

By differentiating the scaled particle[5] approximation for the equation of state, we obtained[16] $\mathscr{W} = \mathscr{W}_3$ of (25), i.e.,

$$\mathscr{W} = (1 - W)^4/(1 + 2W)^2, \tag{64}$$

which also follows from the Percus–Yevick approximation.[6] The corresponding first moment[7] $\overline{F}_1$ of the Wertheim–Thiele solution of the PY integral equation[6] gives

$$N = \frac{2a}{b}\, \frac{6W}{1 + 2W}\left(1 - \frac{W}{5} + \frac{W^2}{10}\right). \tag{65}$$

For $b = 2a$ and the unrealizable value $w = 1$, it follows from the closed forms (64) and (65) that $\mathscr{W} = 0$ and $N = 9/5$; then as shown in the following, $\gamma = \gamma'$. See (2:27)ff for virial expansions in powers of W. The physically realizable range corresponds to $W < W_d \approx 0.63$, and although the range of validity of PY theory is more restricted, the limiting behavior of the closed forms provides a useful check. The simplicity of (64) and (65) obviates detailed computations, and their relations to the rigorous one-dimensional results (29) emphasize the essential physics.

Solving (10) to the required accuracy in terms of the a_n and T's and D's of (19)–(21), and $\mathscr{D} = 1 + \mathscr{B}(1 - w)/3$, we obtain

$$cP_0 = -w(\mathscr{C} - x^2 U_0 + ix^3 \mathscr{C}^2\, \mathscr{W}/3);$$

$$U_0 = T_0 - \mathscr{C}\, D_0 + \frac{\mathscr{C}^2 N}{3} - \frac{\mathscr{C}\,\mathscr{B}\, N\eta^2}{9\,\mathscr{D}},$$

$$T_0 - \mathscr{C}\, D_0 = \tfrac{1}{15}[\mathscr{C}(4 - \eta'^2 - 5\,\mathscr{C}) + \mathscr{B}\,\eta'^2], \tag{66}$$

$$cP_1 = \eta^2 w[\mathscr{B} - x^2 U_1 - ix^3 \mathscr{B}^2\, \mathscr{W}/9\,\mathscr{D}]/\mathscr{D};$$

$$\begin{aligned} U_1 &= \left(\frac{T_1(3 + \mathscr{B})}{3} - \mathscr{B}\, D_1 - \frac{\mathscr{B}^2 N}{9}\right)\frac{1}{\mathscr{D}} + \frac{\mathscr{B}\,\mathscr{C}\, N}{9} \\ &\quad - \frac{\eta^2 \mathscr{B}^2 2}{15}\left[\frac{N}{3\,\mathscr{D}} + \frac{2w}{5}\left(\frac{3 + 2B_1}{3 + 2B'}\right)\right], \end{aligned}$$

$$T_1[(3 + \mathscr{B})/3] - \mathscr{B}\, D_1 = \tfrac{1}{5}[\mathscr{B}(\mathscr{B} + 1) + \mathscr{C}], \tag{67}$$

$$cP_2 = \eta^4 w x^2 U_2, \quad U_2 = \mathscr{B}\,\frac{2}{15}\left(\frac{3 + 2B_1}{3 + 2B'}\right). \tag{68}$$

We proceed as before for (47)–(49). We have

$$\begin{aligned} \eta_c^2 &= \frac{x^2 w}{B_1}\left(-U_0 + \frac{U_1\,\eta_1^2}{\mathscr{D}} - U_2\,\eta_1^4\right) \\ &= \frac{x^2 w}{B_1}\left(A_0 + \frac{A_1}{\mathscr{D}}\,\eta_1^2 + A_2\,\eta_1^4\right), \end{aligned}$$

$$15A_0 = \mathscr{C}\,[\eta'^2 - 4 - 5\,\mathscr{C}(N - 1)] - \mathscr{B}\,\eta'^2,$$

$$45A_1 = \{9[\mathscr{B}(\mathscr{B} + 1) + \mathscr{C}] - 5\,\mathscr{B}^2 N\}/\mathscr{D} + 10\mathscr{C}\,\mathscr{B}\, N,$$

$$15A_2 = -\frac{2N\,\mathscr{B}^2}{3\,\mathscr{D}^2} - \mathscr{B}\,\frac{2}{5}\,\frac{(3 + 2B_1)^2}{(3 + 2B')}, \tag{69}$$

$$\begin{aligned} C_c &= x^2 w\left(-U_0 + \frac{\mathscr{B}\,\eta'^2}{15}\, C_1 + \frac{\mathscr{C}\,\eta_1^2}{5\,\mathscr{D}}\right) \\ &= (x^2 w\,\mathscr{C}/15)[\eta'^2 - 4 - 5\,\mathscr{C}(N - 1) + \mathscr{B}\,\eta'^2 w \\ &\quad + (\eta_1^2/3\,\mathscr{D})(5\,\mathscr{B}\, N + 9)], \end{aligned} \tag{70}$$

$$B_c = x^2w\Big(-\frac{U_1}{\mathscr{D}} + \frac{\mathscr{B}\,\eta'^2 B_1}{15} + \frac{\mathscr{C}}{5\,\mathscr{D}} + \eta_1^2 U_2\Big)$$
$$= \frac{x^2 w\,\mathscr{B}}{15}\Big(\frac{-3(\mathscr{B}+1) + \mathscr{C}(1-w) + \mathscr{B}\,N(5+2\eta_1^2)/3}{\mathscr{D}^2}$$
$$-\frac{5\,\mathscr{C}\,N}{3\,\mathscr{D}} + \eta'^2 B_1 + \eta_1^2\,\frac{2}{5}\,\frac{(3+2B_1)^2}{(3+2B')}\Big). \qquad (71)$$

The set (69)–(71) satisfies $\eta_c^2 B_1 = C_c - \eta_1^2 B_c$. For the unrealizable limit $W = w \to 1$, we have $\mathscr{W} = \mathscr{W}_3 = 0$ and $N = 9/5$, so that $\gamma \to \gamma'$.

The original equation for η^2 obtained from (8) by proceeding as for (35) is a quadratic in η^2,

$$\eta^2 = 1 + w[\mathscr{C} + x^2 A_0 + (ix^3/3)\mathscr{C}^2\mathscr{W}]$$
$$-\frac{w}{\mathscr{D}}\Big(\mathscr{B} - x^2 A_1 - \frac{ix^3\mathscr{B}^2}{9\,\mathscr{D}}\,\mathscr{W}\Big)\eta^2 + wx^2 A_2\eta^4 \qquad (72)$$

corresponding to the analog of (53) for $m = 3$.

For the one-parameter case $B' = 1$, we have

$$\eta_c^2 = C_c = x^2 w\,\mathscr{C}^2(6+3w-5N)/15, \qquad (73)$$

$$\eta_1^2 = C_1 = 1 + w\,\mathscr{C},\quad \eta_s^2 = C_s = x^2 w\,\mathscr{W}\,\mathscr{C}^2/3. \quad (74)$$

Similarly if $C' = 1$, then

$$B_c = -\eta_c^2 B_1^2 = \frac{x^2 w\,\mathscr{B}}{15}\Big(\frac{-3B' + \mathscr{B}\,N(5+2\eta_1^2)/3}{\mathscr{D}^2}$$
$$+\frac{B_1}{B'} + \eta_1^2\,\frac{2}{5}\,\frac{(3+2B_1)^2}{(3+2B')}\Big), \qquad (75)$$

$$B_1 = \frac{1}{\eta_1^2} = 1 + \frac{\mathscr{B}\,w}{\mathscr{D}},$$

$$B_s = -\eta_s^2 B_1^2 = \frac{-x^3 w\,\mathscr{W}\,\mathscr{B}^2}{9\,\mathscr{D}^2}, \qquad (76)$$

where $\mathscr{D} = 1 + \mathscr{B}(1-w)/3$.

If we introduce $1/B' = V'$ and $1/B_1 = V_1$, then the resulting form of η_1^2 equals

$$\eta_1^2 = V_1 = 1 + \frac{\mathscr{V}\,w}{\mathscr{D}_v};\quad \mathscr{V} = V' - 1,$$

$$\mathscr{D}_v = 1 + \frac{\mathscr{V}(1-w)2}{3}. \qquad (77)$$

For $m = \{1,2,3\}$ we have $\mathscr{D}_v = 1 + \mathscr{V}(1-w)(1-m^{-1})$ as compared to $\mathscr{D} = 1 + \mathscr{B}(1-w)m^{-1}$. We also have $B_1\,\mathscr{D} = \mathscr{D}_v/V'$, $\mathscr{B}/B_1\,\mathscr{D} = -\mathscr{V}/\mathscr{D}_v$, so that

$$\eta_s^2 = -B_s/B_1^2 = x^3 w\,\mathscr{W}\,\mathscr{V}^2/9\,\mathscr{D}_v^2, \qquad (78)$$

$$\eta_c^2 = -\frac{B_c}{B_1^2} = \frac{x^2 w\,\mathscr{V}}{15}\Big(\frac{-3 + \mathscr{V}\,N(5+2\,\eta_1^2)/3}{\mathscr{D}_v^2}$$
$$+ V_1 + \eta_1^2\,\frac{2}{5}\,\frac{(3V_1+2)^2}{3V'+2}\Big). \qquad (79)$$

For the simplest acoustic problems, $V' = 1/B'$ is a particle's relative mass density. The electromagnetic two-parameter and one-parameter[3] problems for spheres give different forms than (69)–(79).

For comparison with (73) and (74) to $O(\mathscr{B}^2)$, we have from (75) and (76),

$$B_c \approx -\eta_c^2 \approx -x^2 w\,\mathscr{B}^2(6+3w-5N)7/(15)^2, \qquad (80)$$
$$B_1 \approx 1 + w\,\mathscr{B} - w(1-w)\mathscr{B}^2/2,$$
$$\eta_1^2 \approx 1 - w\,\mathscr{B} + w(1+2w)\mathscr{B}^2/3, \qquad (81)$$
$$B_s \approx -\eta_s^2 \approx -x^3 w\,\mathscr{W}\,\mathscr{B}^2/9. \qquad (82)$$

Thus, to $O(\mathscr{B}^2)$ the statistical packing function in B_c is the same as in C_c of (73) and (3:44).

For $b = 2a$, we define

$$\mathscr{P} = \frac{w(6+3w-5N)}{6} = \frac{w(2-5w+4w^2-w^3)}{2(1+2w)}$$
$$= \frac{w(2-w)(1-w)^2}{2(1+2w)} \qquad (83)$$

as the analog of (40). The relation of $\mathscr{P}$ to $\mathscr{W}_3$ of (25) is indicated by $(1-w)^2/(1+2w) = \mathscr{W}_3^{1/2}$. The functions $\mathscr{W}$ and $\mathscr{P}/w$ decrease monotone from 1 to 0 as w increases from 0 to 1. The product[2,18] $\mathscr{S} = w\,\mathscr{W}$ has a maximum at $w \approx 0.13$, and $\mathscr{P}$ has a maximum at $w \approx 0.22$.

V. COLLECTIVE FORMS

The results (35), (53), and (72) correspond to (8), i.e., to $\eta^2 = 1 - c\,\Sigma P_m$ expressed as

$$\eta^2 = 1 + w(\mathscr{C} + x^2 A_0 + ix^m d_m\,\mathscr{C}^2\,\mathscr{W})$$
$$-\frac{\eta^2 w}{\mathscr{D}}\Big(\mathscr{B} - x^2 A_1 - ix^m d_m\,\frac{\mathscr{B}^2\mathscr{W}}{m\,\mathscr{D}}\Big) + \eta^4 w x^2 A_2, \qquad (84)$$

with $A_2 = 0$ for $m = 1$. The solution of the matrix equation (10) for the P_n isolated corresponding $x^2 U_n(\eta^2)/c$ terms; the regrouping of the sum of U's in (8) in order to make the dependence on η^2 explicit, introduced the coefficients A_n. The explicit versions are special cases of

$$\eta^2 \approx \eta_1^2 + \frac{wx^2}{B_1}\Big(A_0 + \frac{A_1\,\eta_1^2}{\mathscr{D}} + A_2\,\eta_1^4\Big)$$
$$+\frac{ix^m d_m w\,\mathscr{W}}{B_1}\Big(\mathscr{C}^2 + \frac{\mathscr{B}^2}{m\,\mathscr{D}^2}\,\eta_1^2\Big), \qquad (85)$$

with coefficients defined in (32), (50), and (69).

Similarly (33), (51), and (70) are covered by

$$C \approx C_1 + x^2 w\Big(-U_0 + \frac{\eta'^2\,\mathscr{B}\,C_1}{m(m+2)} + \frac{\mathscr{C}\,\eta_1^2}{(m+2)\mathscr{D}}\Big)$$
$$+ ix^m d_m\,\mathscr{C}^2 w\,\mathscr{W}, \qquad (86)$$

and (34), (52), and (71) by

$$B \approx B_1 + x^2 w\Big(-\frac{U_1}{\mathscr{D}} + \frac{\mathscr{B}\,B_1\eta'^2}{m(m+2)} + \frac{\mathscr{C}}{(m+2)\mathscr{D}} + \eta_1^2\,U_2\Big)$$
$$- ix^m\,\frac{d_m}{m}\,\frac{\mathscr{B}^2}{\mathscr{D}^2}\,w\,\mathscr{W}, \qquad (87)$$

with the U_n defined in (30), (31), (47)–(49), and (66)–(68), for $m = 1,2,3$.

VI. RIGID PARTICLES

We derive the analogous results for η^2 for distributions of rigid spheres and cylinders as the limits of the two-parameter functions for $C' \to 0$ and $B' \to 0$. We consider spheres first, because explicit closed forms in W are available for the

required packing functions $\mathscr{W}$ and N. Then we consider cylinders and use the development to infer an additional property of the implicit function $M(W)$.

For rigid spheres we substitute $\mathscr{C} = -1$, $\mathscr{B} = -1$, and $\mathscr{D} = 1 + \mathscr{B}(1-w)/3 = (2+w)/3$ in (69) to obtain

$$A_0 = \frac{9-5N}{15}, \quad A_1 = -\frac{27-5N(1+2w)}{45(2+w)},$$
$$A_2 = \frac{2[(10-w)^2 - 45N]}{[15(2+w)]^2}. \tag{88}$$

Although these vanish for $W = w = 1$ (for which case $N = 9/5$), we show that for $W = w$ each coefficient contains $1-w$ as a factor; this enables us to use the same iteration procedure in (72) as for the two-parameter case even for the unrealizable limit of $w \to 1$.

Substituting (88) into (72) for $\mathscr{C} = \mathscr{B} = -1$, we have

$$\eta^2 = 1 - w + \frac{3w}{2+w}\eta^2 + x^2 w\left(A_0 + \frac{3A_1}{2+w}\eta^2 + A_2\eta^4\right)$$
$$+ \frac{iwx^3\mathscr{W}}{3}\left(1 + \frac{3\eta^2}{(2+w)^2}\right). \tag{89}$$

Solving by iteration yields the leading term of the real part

$$\eta_1^2 = (2+w)/2 = 1 + w/2, \tag{90}$$

as well as the leading term of the imaginary part

$$\eta_s^2 = \frac{x^3 w\,\mathscr{W}(7+2w)}{12(1-w)}; \quad \mathscr{W} = \frac{(1-W)^4}{(1+2W)^2}, \quad W \geqslant w, \tag{91}$$

and the x^2 correction to the real part

$$\eta_c^2 = x^2\frac{w(2+w)}{2(1-w)}\left(A_0 + A_1\frac{3}{2} + A_2\frac{(2+w)^2}{4}\right). \tag{92}$$

The leading terms follow directly from (23) and (26), or from the closed form (2:74); the factor 1/3 in (2:75) should be replaced by 1/12. The physically realizable domain corresponds to $W < W_d \approx 0.63$, and $w \leqslant W$. For $W = w$, (91) has a maximum at $w \approx 0.16$.

It is clear for (91), that even for $W = w \to 1$, the iteration procedure is valid: $1-w$ is a factor of $\mathscr{W}$, and $\mathscr{W}/(1-w)$ vanishes as $O(1-w)^3$. Similarly for (92), for $W = w$ in N of (65) we may factor $1-w$ from the coefficients:

$$A_n = (1-w)\mathscr{A}_n;$$
$$\mathscr{A}_0 = \frac{3-w+w^2}{5(1+2w)}, \quad \mathscr{A}_1 = -\frac{9-w+w^2}{15(2+w)},$$
$$\mathscr{A}_2 = \frac{2(20+2w+5w^2)}{45(2+w)^2(1+2w)}. \tag{93}$$

Substituting into (92) we obtain

$$\eta_c^2 = \frac{x^2 w(2+w)}{180}\left(\frac{74-16w+23w^2}{1+2w} - \frac{9(9-w+w^2)}{2+w}\right)$$
$$= \frac{x^2 w(1-w)}{180(1+2w)}(67 - 44w - 5w^2), \tag{94}$$

which has a maximum at $w \approx 0.28$ and vanishes in the unrealizable limit $w \to 1$. We use these limiting results as guides for the analogous case of cylinders.

For cylinders, we substitute $\mathscr{C} = \mathscr{B} = -1$ and $\mathscr{D} = 1 + \mathscr{B}(1-w)/2 = (1+w)/2$ in (50) to obtain

$$A_0 = \frac{3-4M}{8}, \quad A_1 = \frac{(3-4M)-8}{8(1+w)} + \frac{(1-\mathscr{W})}{2},$$
$$A_2 = \frac{\mathscr{W}}{4(1+w)^2}, \tag{95}$$

which vanish for $W = w = 1$, if we use $\mathscr{W} = \mathscr{W}_2 = 0$ and $M = 3/4$. Substituting into (53) yields

$$\eta^2 = 1 - w + \frac{2w\eta^2}{1+w} + x^2 w\left(A_0 + \frac{2A_1\eta^2}{1+w} + A_2\eta^4\right)$$
$$+ \frac{i\pi x^2 w}{4}\left(1 + \frac{2\eta^2}{(1+w)^2}\right), \tag{96}$$

and iteration gives

$$\eta_1^2 = 1 + w, \tag{97}$$

$$\eta_s^2 = \frac{\pi x^2 w\,\mathscr{W}(3+w)}{4(1-w)}, \quad \mathscr{W} = \frac{(1-W)^3}{1+W}, \tag{98}$$

as obtained directly from (23) and (26), or from the closed form (2:74); see (2:76). In addition, the correction to η_1^2 is given by

$$\eta_c^2 = x^2 w\,[(1+w)/(1-w)]\,[A_0 + 2A_1 + A_2(1+w)^2]. \tag{99}$$

The physically realizable domain corresponds to $W < W_d \approx 0.84$, and $w \leqslant W$.

For (98) and $W = w$, we factor $1-w$ from $\mathscr{W}(w)$, and the result for η_s^2 vanishes for the unrealizable $w = 1$. For (99) however, although the individual A_n vanish for the limiting values $\mathscr{W} = 0$ and $M = 3/4$, this does not insure that η_c^2 vanishes. To eliminate a nonvanishing residue, we require that for large w,

$$3 - 4M = \frac{(1-w)8 + O(1-w)^n}{3+w}, \tag{100}$$

with $n > 1$. This inference, based on the behavior of η_c^2 for spheres in terms of the known closed forms for N and $\mathscr{W}_3$, may facilitate development of a corresponding closed form for M to use with $\mathscr{W}_2$.

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